On the use of a hybrid wave based-statistical energy approach for the analysis of vibro-acoustic systems in the mid-frequency range

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Abstract

Most of the currently available numerical prediction techniques for the analysis of steady-state dynamic vibro-acoustic problems can be classified as being either deterministic or statistical approaches. The Finite Element Method (FEM), the most popular deterministic technique, is limited to the low-frequency range due to its sensitivity to interpolation and pollution errors. The statistical methods, of which the Statistical Energy Analysis (SEA) is most know, are limited to the high-frequency range due to their underlying assumptions. Between the low- and high-frequency ranges there is a relatively wide mid-frequency-range, in which some of the structural subsystems fulfill the requirements for the statistical approach and some others do not (yet). Recently, a hybrid deterministic-statistical framework which combines FE and SEA models has been developed by Shorter and Langley. However, the computational load associated with the FE models still limits the use of this method. In this paper, a hybrid framework is proposed which couples Trefftz-based deterministic models with statistical SEA models. The framework is used to couple SEA models with the recently developed Wave Based Method (WBM) of which the computational efficiency can be exploited to reduce the computational load as compared to the hybrid FE-SEA.

1 Introduction

Most of the current numerical prediction techniques for steady-state dynamic analysis of vibro-acoustic systems can be classified as either deterministic or statistical approaches.

The use of deterministic methods, like the standard Finite Element Method (FEM) [1], is limited to the so-called low-frequency range. The reason for this is twofold. On the one hand, the computational effort of the deterministic techniques typically increases exponentially with frequency, due to an increase in the spatial variation of the dynamic field variables [2]. On the other hand, the response of a system becomes more and more sensitive to small perturbations of its geometrical and material properties as the frequency increases, which introduces a significant level of scatter on the response of nominally identical systems [3].

The statistical methods, like the Statistical Energy Analysis (SEA) [4] are based on energy considerations. They are typically used for the prediction of the mean and variance of the dynamical response of an ensemble of nominally identical systems. However, the underlying assumptions, e.g. a high modal overlap and modal density, limit their use to high-frequency applications.

In-between the low- and the high-frequency ranges there is a relative wide mid-frequency gap, for which currently no mature simulation techniques are available. Recently, an alternative deterministic modeling approach, the so-called Wave Based Method (WBM) [5], has been proposed. This indirect Trefftz method uses wave functions, which are exact solutions of the governing differential equations to describe the dynamic

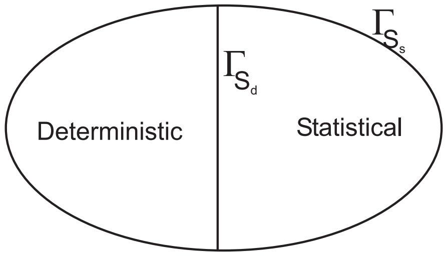

Figure 1: General deterministic-statistical problem

response variables. As a result, the size of the numerical models and the associated computational efforts are substantially lower. The WBM thus stretches the low-frequency range.

The mid-frequency dynamic behavior of many systems, however, exhibits a mixed character in that a number of subsystems have a high modal density and overlap such that they already fulfill the requirements for the application of a statistical description, while some parts of the system still have a deterministic behavior. In view of this mixed dynamic behavior, Langley and co-workers have recently developed a hybrid deterministic-statistical modeling framework [6, 7]. By successfully coupling an FE model to a set of SEA subsystems an efficient and meaningful mid-frequency dynamic model is obtained. The use of the FEM to model the deterministic components in this approach entails however a number of drawbacks, which are foremost related to the FEM's sensitivity to numerical pollution errors and its subsequent high computational cost.

Given its favorable convergence properties, the embedding of the WBM in a similar hybrid WB-SEA framework presents an attractive combination. Due to the indirect nature of the WBM, this method can however not be incorporated readily into the hybrid framework as proposed by Shorter and Langley. In this paper, a suitable extension of the framework is put forward in order to realize this goal. The intrinsic efficiency of the WBM and the ability to represent subsystems with a high-frequency behavior with the SEA result in a method which is able to tackle the mid-frequency region in an efficient and meaningful way.

First, the basic principles of the hybrid approach, the WBM and SEA are briefly explained. In a second part, the WBM for acoustics is combined with SEA for structural vibrations in order to achieve a hybrid WB-SEA method which is able to tackle vibro-acoustic problems. The performance of the newly developed technique is demonstrated by means of the numerical analysis of the mid-frequency behavior of a statistical structure which is mounted on top of a deterministic acoustic cavity. The final section of the paper describes some conclusions and future work.

2 Basic principles

This section describes the basic principles of the different numerical prediction techniques used in this paper. First the hybrid approach is reviewed independently of the problem type(s) and deterministic method considered. The second part describes the basic concepts of the WBM for 3D acoustic problems. Finally, the very basics of SEA are recalled.

2.1 Hybrid approach

This section reviews the hybrid approach as presented by Shorter and Langley in [6] in a way which is independent of the type of deterministic method used. The novelty of this paper lies in the use of an interface grid, the calculation of the coupling matrices and the evaluation of the power balance and will be detailed and applied for the case of a vibro-acoustic system in section 3. Based on the properties of the considered meth-

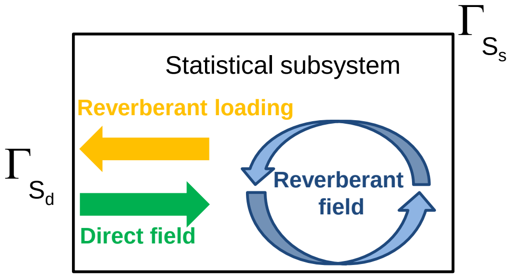

Figure 2: Response of statistical subsystem

ods, the system can be divided into subsystems with deterministic behavior and subsystems with statistical behavior. Figure 1 illustrates a problem consisting of two subsystems.

The uncoupled matrix equation of a deterministic subsystem can be written as:

$$\boldsymbol{A}\boldsymbol{w} = \boldsymbol{b}, \tag{1}$$

with $\boldsymbol{A}$ the system matrix, $\boldsymbol{w}$ the vector of unknowns and $\boldsymbol{b}$ the load vector.

The boundary Γ_S of a statistical subsystem can be divided into a deterministic boundary Γ_{S_d} which is perfectly known, and a boundary Γ_{S_s} which has a statistical behavior. According to [6] the response of the statistical subsystem can be written as a superposition of a 'direct-field' and a 'reverberant field', as illustrated in figure 2. The direct field describes the outgoing field as a result of the prescribed deformation of the deterministic boundary in the absence of the statistical boundary. The direct field thus describes the outgoing field from the deterministic boundary into a (semi-)infinite domain. The reverberant field satisfies blocked boundary conditions across the deterministic boundary and the prescribed boundary conditions across the statistical boundary, when added to the direct field. The diffuse field reciprocity relation [8] allows to calculate the ensemble average of the cross-spectrum $\boldsymbol{S}_{ff}^{rev}$ of the forces on the deterministic boundary of the statistical subsystem caused by the reverberant field as a function of the subsystem average vibrational energy E:

$$\langle \boldsymbol{S}_{ff}^{rev} \rangle = \frac{4E}{\pi\omega n} \mathrm{Im}\left\{\boldsymbol{D}_{dir}\right\}, \tag{2}$$

with $\boldsymbol{D}_{dir}$ the direct field stiffness matrix, ω the circular frequency, n the modal density and $\langle\bullet\rangle$ the ensemble average. If we introduce interface degrees of freedom $\boldsymbol{q}_f$ describing the deformation of the deterministic boundary, the uncoupled equations of motion of a statistical subsystem can be written as:

$$\boldsymbol{D}_{dir}\boldsymbol{q}_f = \boldsymbol{f} + \boldsymbol{f}_{rev,s}, \tag{3}$$

with $\boldsymbol{f}$ the generalized forces and $\boldsymbol{f}_{rev,s}$ the so-called blocked reverberant force on the connection degrees of freedom.

For the sake of simplicity of notation, consider a system consisting of one deterministic and one statistical subsystem. The described methodology can easily be extended to larger systems. Combining the uncoupled equations (1-3) into a coupled matrix equation results in:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{C}_{as} \\ \boldsymbol{C}_{sa} & \boldsymbol{D}_{dir} \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{q}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{f}_{rev,s} \end{bmatrix}, \tag{4}$$

with $\boldsymbol{C}_{as}$ and $\boldsymbol{C}_{sa}$ the coupling matrices between the interface dofs $\boldsymbol{q}_f$ and the degrees of freedom $\boldsymbol{w}$ of the model of the deterministic component. These coupling matrices will be specified for a vibro-acoustic coupling in section 3. Equation (4) can be written as:

$$\boldsymbol{D}_{tot}\boldsymbol{q} = \boldsymbol{f}_{ext} + \boldsymbol{f}_{rev}, \tag{5}$$

with $\boldsymbol{D}_{tot}$ the total system matrix, $\boldsymbol{q}$ the vector of degrees of freedom, $\boldsymbol{f}_{ext}$ the external load vector and $\boldsymbol{f}_{rev}$ the reverberant field load resulting from the random boundaries of the statistical subsystem. Rewriting this equation in cross-spectral form and averaging over an ensemble of statistical boundaries gives:

$$\langle \boldsymbol{S}_{qq} \rangle = \langle \boldsymbol{q}\boldsymbol{q}^H \rangle = \boldsymbol{D}_{tot}^{-1} \langle \boldsymbol{S}_{ff} \rangle \boldsymbol{D}_{tot}^{-H}, \tag{6}$$

with $\bullet^H$ the complex conjugate transpose and with:

$$\langle \boldsymbol{S}_{ff} \rangle = \boldsymbol{S}_{ff}^{ext} + \boldsymbol{f}_{ext} \langle \boldsymbol{f}_{rev}^H \rangle + \langle \boldsymbol{f}_{rev} \rangle \boldsymbol{f}^H + \langle \boldsymbol{f}_{rev} \boldsymbol{f}_{rev}^H \rangle. \tag{7}$$

From the statistics of the blocked reverberant force it follows that $\langle \boldsymbol{f}_{rev} \rangle = 0$ and as a result:

$$\langle \boldsymbol{S}_{ff} \rangle = \boldsymbol{S}_{ff}^{ext} + \langle \boldsymbol{S}_{ff}^{rev} \rangle. \tag{8}$$

Substituting equation (8) in (6) yields:

$$\langle \boldsymbol{S}_{qq} \rangle = \boldsymbol{D}_{tot}^{-1} \boldsymbol{S}_{ff}^{ext} \boldsymbol{D}_{tot}^{-H} + \boldsymbol{D}_{tot}^{-1} \langle \boldsymbol{S}_{ff}^{rev} \rangle \boldsymbol{D}_{tot}^{-H}, \tag{9}$$

in which we define the first term as S_{qq}^{dir} and the second as $\langle S_{qq}^{rev} \rangle$. When combining equations (9) and (2), the only remaining unknown is the vibrational energy E of the statistical subsystem. This can be retrieved by considering the power balance of the statistical subsystem:

$$P_{in}^{dir} = P_{out}^{rev} + P_{diss}, \tag{10}$$

with P_{in}^{dir} the direct field incoming power, P_{out}^{rev} the power leaving the reverberant field into another subsystem and P_{diss} the dissipated power. The first two quantities can be rewritten as a function of the total system matrix $\boldsymbol{D}_{tot}$ and the cross-spectra of the unknowns $\boldsymbol{S}_{qq}^{dir}$ and $\langle \boldsymbol{S}_{qq}^{rev} \rangle$, which in turn depend on the vibrational energy E of the statistical subsystem (section 3). The ensemble average dissipated power P_{diss} can also be written as a function of E. Solving the power balance equation (10) leads to the vibrational energy E of the statistical subsystem and in turn to the ensemble average of the cross-spectrum of the unknowns $\langle \boldsymbol{S}_{qq} \rangle$.

2.2 Wave Based Method for 3D acoustic problems

2.2.1 Problem definition

The steady-state pressure field p at any position $\boldsymbol{r}$, induced by a time-harmonic external point source excitation q with circular frequency ω, located at $\boldsymbol{r}_q$ in the three-dimensional cavity domain V of a bounded acoustic system, is governed by the inhomogeneous Helmholtz equation:

$$\nabla^2 p(\boldsymbol{r}) + k_a^2 p(\boldsymbol{r}) = -j\rho_a \omega q \cdot \delta(\boldsymbol{r}, \boldsymbol{r}_q), \tag{11}$$

with $\nabla^2 \bullet = \frac{\partial^2 \bullet}{\partial x^2} + \frac{\partial^2 \bullet}{\partial y^2} + + \frac{\partial^2 \bullet}{\partial z^2}$, $k_a = \frac{\omega}{c_a}$ the acoustic wavenumber, ρ_a the density and c_a the speed of sound of the fluid.

Since the Helmholtz equation is a second-order differential equation, one boundary condition needs to be specified at each boundary location. The boundary Γ_a can be divided into four non-overlapping parts, $\Gamma_a = \Gamma_p \cup \Gamma_v \cup \Gamma_Z\ cup \Gamma_s$. On each of the first three (deterministic) parts of the boundary, Γ_p, Γ_v and Γ_Z, acoustic pressure, acoustic normal velocity or normal impedance boundary conditions are specified:

$$\forall \boldsymbol{r} \in \Gamma_p : \quad R_p(\boldsymbol{r}) = p(\boldsymbol{r}) - \bar{p}(\boldsymbol{r}) = 0, \tag{12}$$

$$\forall \boldsymbol{r} \in \Gamma_v : \quad R_v(\boldsymbol{r}) = \mathcal{L}_v(p(\boldsymbol{r})) - \bar{v}_n(\boldsymbol{r}) = 0, \tag{13}$$

$$\forall \boldsymbol{r} \in \Gamma_Z : \quad R_Z(\boldsymbol{r}) = \mathcal{L}_v(p(\boldsymbol{r})) - \frac{p(\boldsymbol{r})}{\bar{Z}_n(\boldsymbol{r})} = 0, \tag{14}$$

with $R_\bullet(\boldsymbol{r})$ the boundary residual on quantity $\bullet$ and $\bar{p}(\boldsymbol{r})$, $\bar{v}_n(\boldsymbol{r})$ and $\bar{Z}_n(\boldsymbol{r})$ the prescribed fields for acoustic pressure, normal velocity and normal impedance, respectively. $\mathcal{L}_v(\bullet) = \frac{j}{\rho_a \omega} \frac{\partial \bullet}{\partial \boldsymbol{n}}$ is the normal velocity operator. The boundary condition on the interface Γ_s with the statistical subsystem is defined in section 3.

2.2.2 Field variable expansion

The WBM [5], which is based on an indirect Trefftz approach [9], approximates the cavity pressure by a weighted sum of wave functions, which exactly satisfy the Helmholtz equation (11):

$$p(\boldsymbol{r}) \approx \hat{p}(\boldsymbol{r}) = \sum_{a=1}^{n_w} \Phi_a(\boldsymbol{r}) w_a + \hat{p}_q(\boldsymbol{r}) \tag{15}$$

$$= \boldsymbol{\Phi}(\boldsymbol{r}) \cdot \boldsymbol{w} + \hat{p}_q(\boldsymbol{r}) \tag{16}$$

with $\boldsymbol{w}$ a $(n_w \times 1)$ vector of unknown wave function contribution factors w_a, and $\boldsymbol{\Phi}(\boldsymbol{r})$ a $(1 \times n_w)$ vector of wave functions $\Phi_a(\boldsymbol{r})$. $\hat{p}_q(\boldsymbol{r})$ is a particular solution of the inhomogeneous Helmholtz equation:

$$\hat{p}_q(\boldsymbol{r}) = \frac{j\rho\omega q}{4\pi} \frac{e^{-jk_a d(\boldsymbol{r},\boldsymbol{r}_q)}}{d\left(\boldsymbol{r},\boldsymbol{r}_q\right)}, \tag{17}$$

with $d\left(\boldsymbol{r},\boldsymbol{r}_q\right)$ the distance between a point at $\boldsymbol{r}$ and the acoustic source q at location $\boldsymbol{r}_q$.

Desmet [5] proposes to use a superposition of three types of wave functions (the so-called *r*-, *s*- and *t*-set) as basis functions to describe the homogeneous solution for the steady-state acoustic pressure field:

$$\Phi_a(\boldsymbol{r}) = \begin{cases} \Phi_r(\boldsymbol{r}) = \cos\left(k_{xr}x\right)\cos\left(k_{yr}y\right)e^{-jk_{zr}z} \\ \Phi_s(\boldsymbol{r}) = \cos\left(k_{xs}x\right)e^{-jk_{ys}y}\cos\left(k_{zs}z\right) \\ \Phi_t(\boldsymbol{r}) = e^{-jk_{xt}x}\cos\left(k_{yt}y\right)\cos\left(k_{zt}z\right) \end{cases}. \tag{18}$$

In order for the wave functions to exactly satisfy the homogeneous Helmholtz equation (11), the wave number components k_{ij} need to satisfy:

$$\left(k_{xj}\right)^2 + \left(k_{yj}\right)^2 + \left(k_{zj}\right)^2 = k_a^2 \quad \text{with} \quad j = r, s, t. \tag{19}$$

It can be proven that a convergent set of wave functions is obtained if the wave numbers are chosen as follows [5]:

$$\begin{cases} \left(k_{xr}, k_{yr}, k_{zr}\right) = \left(\frac{a_1\pi}{L_x}, \frac{a_2\pi}{L_y}, \pm\sqrt{k_a^2 - \left(\frac{a_1\pi}{L_x}\right)^2 - \left(\frac{a_2\pi}{L_y}\right)^2}\right), & a_1 = 0,1,2\dots \quad a_2 = 0,1,2\dots \\ \left(k_{xs}, k_{ys}, k_{zs}\right) = \left(\frac{a_3\pi}{L_x}, \pm\sqrt{k_a^2 - \left(\frac{a_3\pi}{L_x}\right)^2 - \left(\frac{a_4\pi}{L_z}\right)^2}, \frac{a_4\pi}{L_z}\right), & a_3 = 0,1,2\dots \quad a_4 = 0,1,2\dots \\ \left(k_{xt}, k_{yt}, k_{zt}\right) = \left(\pm\sqrt{k_a^2 - \left(\frac{a_5\pi}{L_y}\right)^2 - \left(\frac{a_6\pi}{L_z}\right)^2}, \frac{a_5\pi}{L_y}, \frac{a_6\pi}{L_z}\right), & a_5 = 0,1,2\dots \quad a_6 = 0,1,2\dots \end{cases} \tag{20}$$

with L_x, L_y and L_z the dimensions of the (preferably) smallest rectangular box enclosing the considered domain, as illustrated in figure 3.

Equation (20) contains an infinite series of wave numbers and, as a result, there is an infinite number of wave functions. This series is truncated by defining an upper bound $n_\bullet$ on each of the parameters $a_\bullet$ through a frequency dependent truncation rule:

$$\frac{n_1}{L_x} = \frac{n_2}{L_y} = \frac{n_3}{L_x} = \frac{n_4}{L_z} = \frac{n_5}{L_y} = \frac{n_6}{L_z} \geq N\frac{k_a}{\pi}, \tag{21}$$

with N the 'truncation factor'. N typically ranges between 1 and 6. Applying this truncation rule results in the use of all wave functions with wavelengths larger than or equal to $1/N$ times the physical wavelength at each frequency of interest.

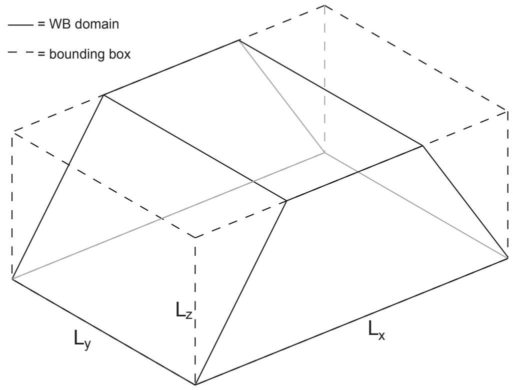

Figure 3: Bounding box enclosing a WB domain

2.2.3 System of equations

The wave functions in equation (18) satisfy the governing dynamic equation, but not necessarily the specified boundary conditions. These conditions are enforced by minimizing the boundary residuals $R_\bullet$ (equations (12-14)) in a weighted residual formulation:

$$-\int_{\Gamma_p} \frac{j}{\rho_a \omega} \frac{\partial \tilde{p}(\boldsymbol{r})}{\partial \boldsymbol{n}} \cdot R_p(\boldsymbol{r}) \mathrm{d}\Omega + \int_{\Gamma_v} \tilde{p}(\boldsymbol{r}) \cdot R_v(\boldsymbol{r}) \mathrm{d}\Omega + \int_{\Gamma_Z} \tilde{p}(\boldsymbol{r}) \cdot R_Z(\boldsymbol{r}) \mathrm{d}\Omega = 0. \tag{22}$$

As commonly used in the FEM, the WBM adopts a Galerkin weighted residual formulation in which the weighting functions $\tilde{p}(\boldsymbol{r})$ are expanded in terms of the same set of wave functions used in the field variable expansion (15). This leads to a matrix equation consisting of n_w algebraic equations in the n_w unknown wave contribution factors, which is typically written as:

$$\boldsymbol{A}\boldsymbol{w} = \boldsymbol{b}. \tag{23}$$

As for the Boundary Element Method (BEM) [10] and in contrast with the FEM, the WBM yields a fully populated matrix, of which the elements are complex and which cannot be decomposed into frequency independent matrices. The big advantage of the WBM is, however, that the system matrices are substantially smaller in comparison with the element based techniques. This property, combined with the fast convergence and the absence of pollution error, makes the WBM a less computationally demanding method for dynamic response calculations, which creates opportunities to tackle problems up to higher frequencies. A more detailed description of the WBM and its properties for different problem types can be found in [5, 11, 12, 13, 14, 15, 16, 17, 18, 19].

2.3 Statistical Energy Analysis

In contrast to the deterministic methods such as the FEM, BEM, WBM, etc. SEA does not pose the problem in terms of a distribution of the field variable throughout the problem domain. SEA tries to calculate space and frequency averaged energy quantities for a statistical ensemble. It originates from the late fifties where Lyon [20] and Smith [21] did the pioneering work and has become a valuable simulation technique for the prediction of the averaged response in large structures like ships, airplanes, busses, etc. It has also found its application for smaller systems, e.g. automotive problems, where it however is restricted to the so-called high-frequency range due to the basic assumptions of this technique. This section briefly reviews the basic equation of SEA. The reader is referred to e.g. the book by Lyon and DeJong [4] for more details.

In a first step, the system is divided into a (small) number of subsystems. Each subsystem is defined as an element of an SEA model corresponding to a substantial energy storage location and should be chosen

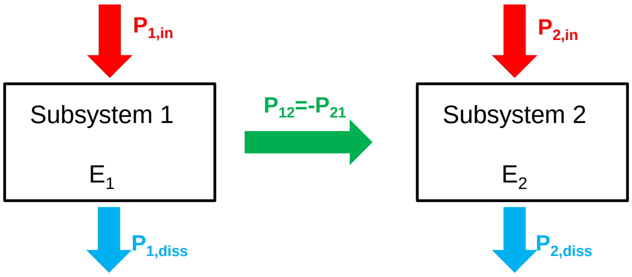

Figure 4: Energy balance of a two-subsystem problem.

according to the principle of similarity [22]. This means that the modes contained in a subsystem should be similar in energetic terms and should have more or less the same order of damping. Two additional hypotheses are made: the modes of a subsystem are equally distributed in the considered frequency interval, and all modes of a subsystem are equally energetic and the modal responses are incoherent.

The global SEA equations of a system are then obtained through an energy balance of each subsystem i, see figure 4:

$$P_{i,in} = P_{i,diss} + \sum_{j \neq i}^{n} P_{ij}, \tag{24}$$

with $P_{i,in}$ the power input from the environment, $P_{i,diss}$ the power dissipated in the subsystem and P_{ij} the power flow from subsystem i to subsystem j.

The time averaged energy flow from subsystem i to subsystem j is considered to be proportional to the difference in modal energy:

$$P_{ij} = \omega \left(\eta_{ij} E_i - \eta_{ji} E_j \right), \tag{25}$$

with E_i the time averaged total energy of subsystem i and η_{ij} the SEA coupling loss factor, which is a measure for the transfer of energy from subsystem i due to coupling with subsystem j that is coupled to subsystem i. The coupling loss factors η_{ij} and η_{ji} are related through the reciprocity relation:

$$n_i \eta_{ij} = n_j \eta_{ji}, \tag{26}$$

with n_i the modal density of subsystem i. The coupling loss factors are most often determined based on the transmission coefficient of a junction. The determination of the transmission coefficient is however far from trivial and mostly based on analytical solutions for (semi-)infinite systems or measurements of (similar) subsystems.

The time averaged dissipated power in subsystem i is:

$$P_{i,diss} = \omega \eta_i E_i = M_i \frac{E}{n_i} \tag{27}$$

with η_i and M_i the damping loss factor and the modal overlap factor of subsystem i.

Rewriting equation (24) in matrix notation with the subsystem modal energies as unknowns leads to:

$$\omega \begin{bmatrix} \left(\eta_1 + \sum_{j\neq 1}^{n} \eta_{1,j}\right) n_1 & -\eta_{12} n_1 & \dots & -\eta_{1n} n_1 \\ \vdots & \ddots & & \vdots \\ -\eta_{n1} n_n & -\eta_{n2} n_n & \dots & \left(\eta_n + \sum_{j\neq n}^{n} \eta_{n,j}\right) n_n \end{bmatrix} \begin{Bmatrix} E_1/n_1 \\ \vdots \\ E_n/n_n \end{Bmatrix} = \begin{Bmatrix} P_1 \\ \vdots \\ P_n \end{Bmatrix}, \tag{28}$$

in which the system matrix is small and symmetric due to the reciprocity relation.

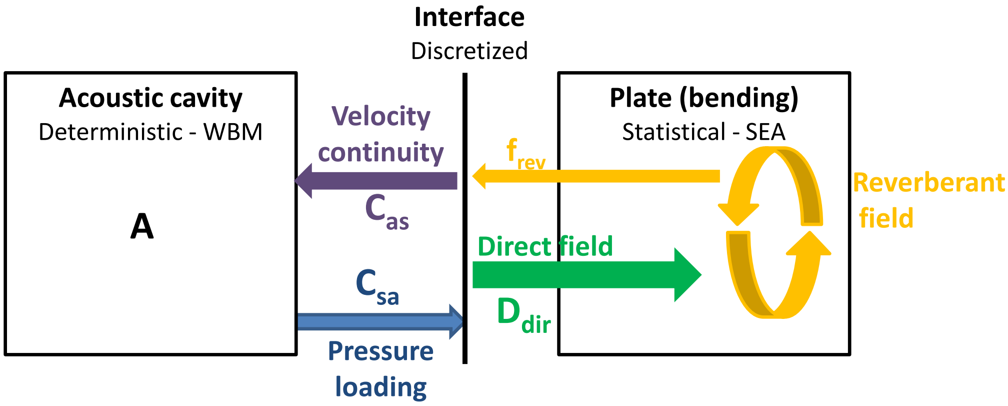

Figure 5: Concept of the vibro-acoustic hybrid WB-SEA

3 Hybrid WB-SEA for vibro-acoustic problems

This section describes the hybrid WB-SEA for a coupled vibro-acoustic problem, with a 3D deterministic acoustic cavity that is coupled to a 2D elastic plate with statistical properties. The hybrid methodology described in section 2.1 is adopted, using a discretized frame with displacement degrees of freedom q_f on the interface between the deterministic and statistical subsystem. This discretized interface frame is introduced in order to be able to use the diffuse reciprocity relationship as defined by Shorter and Langley [6]. Equations (4) and (10) describe the system, but the coupling matrices C_{sa} and C_{as} and the different terms of the power balance equation still need to be specified.

First the coupling of the acoustic load in the deterministic subsystem to the statistical subsystem (C_{sa}) is determined by considering the response of an infinite plate to the pressure load associated with the acoustic wave functions. Secondly, the back-coupling (C_{as}) from the structure to the acoustic cavity is obtained by imposing normal velocity continuity at the interface. Finally, the power balance equation is elaborated. The concept is illustrated in figure 5.

Without loss of generality and for ease of notation, the method will be explained for a system consisting of only one deterministic and one statistical subsystem.

3.1 Acoustic → structure coupling (C_{sa})

As we consider the plate as being statistical, we assume that the edges of the plate are the 'statistical boundary', having statistical properties. The direct field in the plate is defined as the field corresponding to a plate with the same deterministic boundary but considering the statistical boundary absent. In this case, that results in an infinite plate which is loaded on the deterministic interface with the acoustic cavity. We can thus write the displacement of the plate, and, as a result, also the interface degrees of freedom $\boldsymbol{q}_f$, as the response of an infinite plate to the acoustic pressure on the interface and the reverberant field loading:

$$\boldsymbol{q}_f = \int_{\Gamma_{S_d}} p_a(\boldsymbol{r}) G\left(d\left(\boldsymbol{r}_f, \boldsymbol{r}\right)\right) \mathrm{d}\Gamma + \boldsymbol{H}_{dir} \boldsymbol{f}_{rev}, \tag{29}$$

with $G(d)$ the Green's function, $H_{dir,jk} = G\left(d\left(\boldsymbol{r}_j, \boldsymbol{r}_k\right)\right)$ and $\boldsymbol{r}_f$ the location vector of the interface points. For Kirchhoff plate bending the Green's function is defined as:

$$G(d) = -\frac{j}{8k_b^2 D}\left[H_0^{(2)}(k_b d) - H_0^{(2)}(-jk_b d)\right], \tag{30}$$

with k_b the wavenumber, D the flexural rigidity of the plate and $H_0^{(2)}$ the zeroth order Hankel function of the second kind.

Using the acoustic pressure expansion (15) equation (29) becomes:

$$\boldsymbol{q}_f = \int_\Gamma G\left(d\left(\boldsymbol{r}_f, \boldsymbol{r}\right)\right)\left(\boldsymbol{\Phi}(\boldsymbol{r})\boldsymbol{w} + \hat{p}_q(\boldsymbol{r})\right)\mathrm{d}\Gamma + \boldsymbol{H}_{dir}\boldsymbol{f}_{rev}, \tag{31}$$

or if we define $\boldsymbol{D}_{dir} = \boldsymbol{H}_{dir}^{-1}$:

$$\left(-\boldsymbol{D}_{dir}\int_\Gamma G\left(d\left(\boldsymbol{r}_f, \boldsymbol{r}\right)\right)\boldsymbol{\Phi}(\boldsymbol{r})\mathrm{d}\Gamma\right)\boldsymbol{w} + \boldsymbol{D}_{dir}\boldsymbol{q}_f = \boldsymbol{f}_{rev} + \boldsymbol{D}_{dir}\int_\Gamma G\left(d\left(\boldsymbol{r}_f, \boldsymbol{r}\right)\right)\hat{p}_q(\boldsymbol{r})\mathrm{d}\Gamma, \tag{32}$$

$$\boldsymbol{C}_{sa}\boldsymbol{w} + \boldsymbol{D}_{dir}\boldsymbol{q}_f = \boldsymbol{f}_{rev} + \boldsymbol{f}. \tag{33}$$

Equation (33) corresponds to the lowest line in the matrix equation (4), defining the matrices $\boldsymbol{C}_{sa}$ and $\boldsymbol{f}$:

$$\boldsymbol{C}_{sa} = -\boldsymbol{D}_{dir}\int_\Gamma G\left(d\left(\boldsymbol{r}_f, \boldsymbol{r}\right)\right)\boldsymbol{\Phi}(\boldsymbol{r})\mathrm{d}\Gamma, \tag{34}$$

$$\boldsymbol{f} = \boldsymbol{D}_{dir}\int_\Gamma G\left(d\left(\boldsymbol{r}_f, \boldsymbol{r}\right)\right)\hat{p}_q(\boldsymbol{r})\mathrm{d}\Gamma. \tag{35}$$

3.2 Structure → acoustic coupling (C_{as})

The presence of a vibrating structure along the boundary Γ_s of the acoustic cavity acts as a velocity excitation for the acoustic fluid. The corresponding residual,

$$R_{va}(\boldsymbol{r}) = \mathcal{L}_v\left(p(\boldsymbol{r})\right) - j\omega u(\boldsymbol{r}), \tag{36}$$

with $u(\boldsymbol{r})$ the normal displacement field of the plate, gives the following extra term in the weighted residual formulation (22):

$$\int_\Gamma \tilde{p}_a(\boldsymbol{r})\cdot R_{va}(\boldsymbol{r})\mathrm{d}\Gamma = \tilde{\boldsymbol{w}}^T\left[\int_\Gamma \boldsymbol{\Phi}^T(\boldsymbol{r})\mathcal{L}_v\left(\boldsymbol{\Phi}(\boldsymbol{r})\right)\boldsymbol{w}\mathrm{d}\Gamma - \int_\Gamma j\omega\boldsymbol{\Phi}^T(\boldsymbol{r})\boldsymbol{N}(\boldsymbol{r})\boldsymbol{q_f}\mathrm{d}\Gamma\right] \tag{37}$$

$$= \tilde{\boldsymbol{w}}^T\left[\boldsymbol{C}_{aa}\boldsymbol{w} + j\omega\boldsymbol{C}_{as}\boldsymbol{q}_f\right], \tag{38}$$

with $\boldsymbol{N}(\boldsymbol{r})$ a vector containing shape functions interpolating the nodal displacements $\boldsymbol{q}_f$. In the numerical example in this paper, linear shape functions are used. The structural-acoustic coupling matrix $\boldsymbol{C}_{as}$ is found as:

$$\boldsymbol{C}_{as} = \int_\Gamma j\omega\boldsymbol{\Phi}^T(\boldsymbol{r})\boldsymbol{N}(\boldsymbol{r})\mathrm{d}\Gamma. \tag{39}$$

3.3 Power balance equation

By applying the conservation of energy for a steady-state dynamic system, the power balance equation (10) is defined, with the dissipated power P_{diss} as in equation (27).

The time and ensemble averaged input power to the direct field of the statistical subsystem can be written as:

$$P_{in}^{dir} = \frac{\omega}{2}\sum_{jk}\mathrm{Im}\left\{D_{dir,jk}\right\}\langle S_{qq,jk}^{dir}\rangle \tag{40}$$

$$= \frac{\omega}{2}\sum_{jk}\mathrm{Im}\left\{D_{dir,jk}\right\}\boldsymbol{D}_{tot}^{-1}\boldsymbol{S}_{ff}^{ext}\boldsymbol{D}_{tot}^{-H}. \tag{41}$$

The time averaged power leaving the reverberant field of the statistical plate subsystem and going into the acoustic cavity can be calculated by integrating the acoustic intensity over the interface surface:

$$P_{out}^{rev} = \frac{1}{2}\mathrm{Re}\left\{\int_\Gamma p_a(\boldsymbol{r})v_a^*(\boldsymbol{r})\mathrm{d}\Gamma\right\}. \tag{42}$$

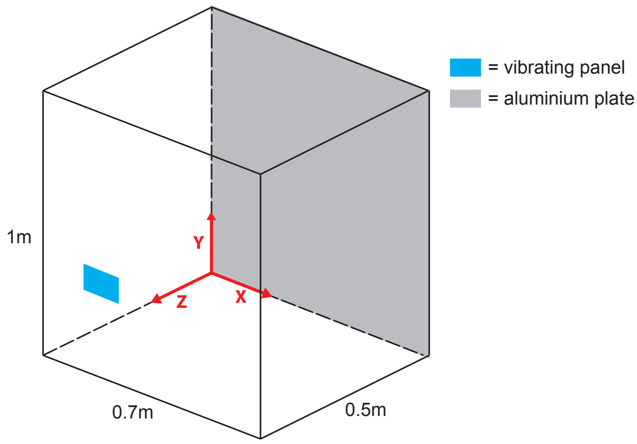

Figure 6: Problem geometry

For ease of notation we omit from now on the position dependency $(\boldsymbol{r})$. Using the field variable expansion (15) and with $\boldsymbol{B}$ a $n_w \times 1$ vector containing the normal derivatives of the wave functions Φ_a and $\boldsymbol{w}_{rev}$ a $n_w \times 1$ vector containing the wave function contribution factors due to the reverberant field loading, this can be written as:

$$P_{out}^{rev} = \frac{1}{2}\mathrm{Re}\left\{\int_\Gamma \left(\boldsymbol{\Phi}^T \boldsymbol{w}_{rev}\right)\left(\frac{j}{\rho_0\omega}\boldsymbol{B}^T \boldsymbol{w}_{rev}\right)^* \mathrm{d}\Gamma\right\}. \tag{43}$$

Using some simple mathematics which we omit here, this equation can be rewritten as:

$$P_{out}^{rev} = \frac{1}{2}\mathrm{Re}\left\{\frac{-j}{\rho_0\omega}\int_\Gamma \sum_{jk} \left(\boldsymbol{\Phi}\boldsymbol{B}^H\right)_{jk}\left(\boldsymbol{w}_{rev}\boldsymbol{w}_{rev}^H\right)_{jk} \mathrm{d}\Gamma\right\} \tag{44}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{-j}{\rho_0\omega}\sum_{jk}\left(\int_\Gamma \left(\boldsymbol{\Phi}\boldsymbol{B}^H\right)_{jk} \mathrm{d}\Gamma\right)\left(\boldsymbol{w}_{rev}\boldsymbol{w}_{rev}^H\right)_{jk}\right\}. \tag{45}$$

Taking into account the ensemble average results in:

$$P_{out}^{rev} = \frac{1}{2}\mathrm{Re}\left\{\frac{-j}{\rho_0\omega}\sum_{jk}\left(\int_\Gamma \left(\boldsymbol{\Phi}\boldsymbol{B}^H\right)_{jk} \mathrm{d}\Gamma\right)\langle \boldsymbol{S}_{qq,jk}^{rev}\rangle\right\} \tag{46}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{-j}{\rho_0\omega}\sum_{jk}\left(\int_\Gamma \left(\boldsymbol{\Phi}\boldsymbol{B}^H\right)_{jk} \mathrm{d}\Gamma\right)\boldsymbol{D}_{tot}^{-1}\langle \boldsymbol{S}_{ff}^{rev}\rangle \boldsymbol{D}_{tot}^{-H}\right\}, \tag{47}$$

with $\langle \boldsymbol{S}_{ff}^{rev}\rangle$ as defined in equation (2).

All terms in the power balance (10) have the vibrational energy E of the statistical system as the only unknown quantity. Solving this equation leads to this vibrational energy, which then is used in equation (9) to calculate the spectrum of the wave function contribution factors w and as a result also the spectrum of the pressure field in the acoustic cavity.

4 Numerical example

The hybrid approach described above provides a numerical modeling framework to combine an acoustical WB model with an SEA model for a plate. In order to illustrate this approach, the method is applied to a

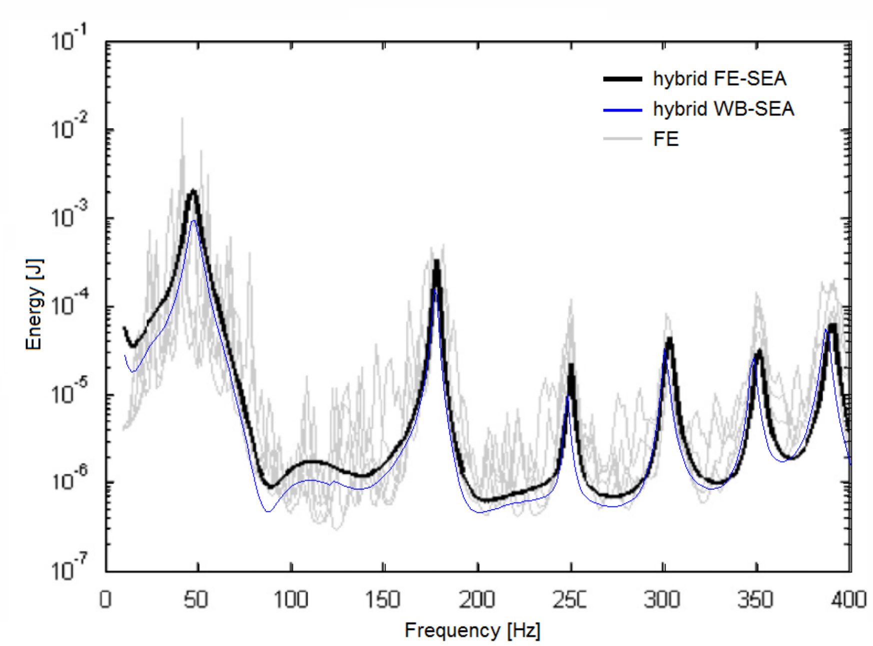

Figure 7: Energy in the plate calculated with hybrid FE-SEA (thick black), hybrid WB-SEA (thin blue) and four different FE models (grey).

simple validation example taken from literature [7]. Consider a $0.7 \times 1 \times 0.5m$ acoustic cavity filled with air (density $\rho = 1.225\frac{kg}{m^3}$, sound speed $c = 340\frac{m}{s}$, loss factor 0.01), as shown in figure 6. At one side a $1mm$ thick aluminium plate is connected (density $\rho = 2700\frac{kg}{m^3}$, Youngs modulus $E = 70GPa$, damping loss factor $\eta = 0.01$, modal density $n = 0.227$ modes$/Hz$). The other walls are considered to be acoustically rigid. The system is excited by a vibrating panel of $8 \cdot 10^{-4}m^2$ on the front wall with its center at the point $(0.214; 0.252; 0.5)$.

Figure 7 compares the results obtained by three different methods in a frequency range of 0 to 400 Hz (below 400 Hz: about 90 structural modes and 6 acoustic modes). A hybrid WB-SEA model is constructed in which a WB model with truncation factor $N = 2$ for the acoustic cavity is coupled through 1189 interface nodes ($\approx$ 6 interface elements per wavelength in the plate at 400Hz) to the plate which is considered to be statistical. The results are compared with the results in [7] which contains a hybrid FE-SEA model and five pure FE-calculations, each of which has a different irregular combination of clamped and simply supported boundary conditions around the plate perimeter. A good agreement with the hybrid FE-SEA model is seen. The small discrepancies are expected to be caused by model uncertainties regarding the material properties and the exact source definition. Both hybrid approaches seem to predict well the average of the full FE-calculations with perturbed plate boundary conditions. The cavity modes are clearly visible in the response, the plate modes cannot be seen since the effect of these is averaged in the statistical model.

We conclude from this example that the hybrid WB-SEA gives accurate prediction results. Due to the much smaller WB matrices as compared to the FE matrices and its high computational efficiency, the hybrid WB-SEA emerges as a promising technique for mid-frequency problems. However, further comparison of the hybrid WB-SEA and the hybrid FE-SEA in terms of efficiency is a point of further research.

5 Conclusion

In many applications some parts of the considered system behave deterministically and some statistically. Previously Shorter and Langley [6] presented a hybrid framework to couple the FEM and the SEA. This paper proposes a modification of that hybrid framework to couple a Trefftz-based deterministic method with a statistical method. As a result, the WBM for acoustics, which has proven to be a more accurate and efficient deterministic method as compared to the FEM, can be coupled with SEA for structural dynamics. The developed method is compared on a benchmark case available in literature to the existing hybrid FE-SEA method and various FE calculations. In the near future, the approach will be applied for larger systems consisting of multiple deterministic and statistical subsystems and the efficiency will be assessed in comparison to the available techniques.

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